Closed Forms for Weighted Model Counting

in the Two Variable Fragment

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Introduction

- Weighted First Order Model Counting $\text{WFOMC}(\Phi, n) = \sum_{\omega \models \Phi} w(\omega)$
- **Probabilistic Inference** in many Probabilistic Logic Frameworks can be cast as WFOMC. But WFOMC is **#P**!!!.
- Fragments of First Order Logic (FOL) that admit polynomial time WFOMC are called **Domain Liftable**.
- In this work we revisit the domain liftable fragments of First Order Logic, providing **combinatorial proofs** and **closed for formulas** for WFOMC.

Universally Qauntified Formulas

FOL formula of the form $\forall x \forall y. \Phi(x, y)$
Example: Simple Undirected Graphs

$$\Phi(x, y) = \neg R(x, x) \land (R(x, y) \rightarrow R(y, x))$$

Formula:

$$\sum_{\boldsymbol{k},\boldsymbol{h}} \binom{n}{\boldsymbol{k}} \prod_{0 \leq i \leq j \leq 2^u - 1} \binom{\boldsymbol{k}(i,j)}{\boldsymbol{h}^{ij}} \prod_{0 \leq v \leq 2^b - 1} n_{ijv}^{h_v^{ij}}$$

where $\boldsymbol{k} = (k_0, \ldots, k_{2^u-1})$ is a 2^u-tuple of non-negative integers, $\binom{n}{\boldsymbol{k}}$ is the multinomial coefficient and for every $0 \leq i \leq j \leq 2^u - 1$, $\boldsymbol{h}^{ij} = (h_0^{ij}, \ldots, h_{2^b-1}^{ij})$ is a vector of 2^b integers that sum up to $\boldsymbol{k}(i, j)$.

$$\boldsymbol{k}(i,j) = \begin{cases} \frac{k_i(k_i-1)}{2} & \text{if } i = j \\ k_i k_j & \text{otherwise} \end{cases}$$

Key Ideas:

- k_i represents the number of constants that realise the i^{th} unary assignment, for example, for a constant c, either $R(c, c)$ or $\neg R(c, c)$ is true. Similarly, h_v^{ij} represents the number of pairs of constants which realise the v^{th} binary assignment, for example $\{R(c, d), R(d, c)\}$, $\{R(c, d), \neg R(d, c)\}$ etc.
- $\binom{n}{\boldsymbol{k}} \prod_{0 \leq i \leq j \leq 2^u-1} \binom{\boldsymbol{k}(i,j)}{\boldsymbol{h}^{ij}} \prod_{0 \leq v \leq 2^b-1}$ represent the possible ways of achieving models with k_i constants of i^{th} unary assignments and h_v^{ij} constants of v^{th} binary assignment.
- $\{n_{ijv}\}$ is a 1 iff the pair of constants with i^{th} and j^{th} unary assignment and v^{th} binary assignment are consistent with Φ and 0 otherwise.

Cardinality Constraints

FOL formula in two variable fragment with constraints on the number of true interpretations of a certain predicate.
Example: Simple Undirected Graphs with only l edges.

$$\Phi(x, y) = \neg R(x, x) \land (R(x, y) \rightarrow R(y, x)) \land (|R| = 2l)$$

Formula:

$$\sum_{\boldsymbol{k},\boldsymbol{h} \models \rho} \binom{n}{\boldsymbol{k}} \prod_{0 \leq i \leq j \leq 2^u - 1} \binom{\boldsymbol{k}(i,j)}{\boldsymbol{h}^{ij}} \prod_{0 \leq v \leq 2^b - 1} n_{ijv}^{h_v^{ij}}$$

where ρ is a constraint on the integer tuples $\boldsymbol{k}$ and $\boldsymbol{h}^{ij}$.

Key Ideas:

- $\boldsymbol{k}$ and $\boldsymbol{h}^{ij}$ contain all the cardinality information of each predicate.

Existential Quantifiers

FOL formula of the form: $\forall x \forall y. \Phi(x, y) \land \bigwedge_{i=1}^{q} \forall x \exists y. \Psi_i(x, y)$
Example: Simple Unidrected Graphs with atleast one edge from each node.

$$\forall xy. \neg R(x, x) \land (R(x, y) \rightarrow R(y, x)) \land \forall x \exists y. R(x, y)$$

Formula:

$$\sum_{\boldsymbol{k},\boldsymbol{h} \models \rho} (-1)^{\sum_i \boldsymbol{k}(P_i)} \binom{n}{\boldsymbol{k}} \prod_{0 \leq i \leq j \leq 2^u - 1} \binom{\boldsymbol{k}(i,j)}{\boldsymbol{h}^{ij}} \prod_{0 \leq v \leq 2^b - 1} n_{ijv}^{h_v^{ij}}$$

Key Ideas:

- Principle of Inclusion Exclusion: $e_0 = \sum_{l=0}^{m} (-1)^l s_l$
- $e_0 = \text{FOMC}(\forall x \forall y. \Phi(x, y) \land \bigwedge_{i=1}^{q} \forall x \exists y. \Psi_i(x, y))$
- $s_l = \text{FOMC}(\forall x \forall y. \Phi(x, y) \land \bigwedge_{i=1}^{q} (P_i(x) \rightarrow \neg \Psi_i(x, y)) \land (\sum_i |P_i| = l)$

Counting Quantifiers

Counting Quantifiers are quantifiers of the form $\exists^{=k}$, $\exists^{\leq}$ and $\exists^{\geq}$. Any formula with counting quantifiers can be written as the following model count preserving reduction:

$$\Phi_0 \land \bigwedge_{k=1}^{q} \forall x. (A_k(x) \leftrightarrow \exists^{=m_k} y. R_k(x, y))$$

Example: Simple Undirected k-Regular Graphs

$$\forall xy. \neg R(x, x) \land (R(x, y) \rightarrow R(y, x)) \land \forall x \exists^{=k} y. R(x, y)$$

Key Ideas:

- Expressing the counting quantifiers $\exists^{=k}$ as a set of formulas with existential quantifiers and cardinality constraints.
- Principle of Inclusion Exclusion: $e_0 = \sum_{l=0}^{m} (-1)^l s_l$

Weighted Model Counting

- Weighted First Order Model Counting can be easily obtained by defining a multiplicative real valued weight function on the $(\boldsymbol{k}, \boldsymbol{h})$ vectors.
- Symmetric Weighted Model Counting :

$$w(\boldsymbol{k}, \boldsymbol{h}) = \prod_{P \in \mathcal{L}} w(P)^{(\boldsymbol{k},\boldsymbol{h})(P)} \cdot \bar{w}(P)^{(\boldsymbol{k},\boldsymbol{h})(\neg P)}$$

- Expressing Count Distributions:

$$w(\boldsymbol{k}, \boldsymbol{h}) : (\boldsymbol{k}, \boldsymbol{h}) \rightarrow \text{R}$$

References

1. P. Beame *et al.*, *Symmetric Weighted First-Order Model Counting*, 2015, arXiv: 1412.1505.
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3. S. Malhotra *et al.*, *Weighted Model Counting in FO2 with Cardinality Constraints and Counting Quantifiers: A Closed Form Formula*, 2021, arXiv: 2110.05992.